Altitude Theory

As performed by the
~brilliant~
Middleton bros.

Altitude is one of the fundamental measurements crucial to the successful operation of any flying machine, be it piloted or autonomous. Altitude is typically measured through barometric pressure or air pressure as I'll often call it here (so shoot me – at least it's quicker to type). I might note that the calculations for altitude and airspeed are somewhat related, since both are determined basically by measuring air pressure, and because they share similar equations derived from the same theorems. However, altitude is probably the best place to start if you're just beginning.

The basic set of equations that describe the features of our atmosphere at various altitudes are contained in what is called the International Standard Atmosphere (ISA) model – there is also a U.S. Standard Atmosphere model, and both are very similar. This model is theoretically updated every so often, but the last time scientists did so, to my understanding, was in 1976 for the US version and I don't know when for the International version. Do a search for these on Google or wherever and you'll come up with a bunch of Java calculators and summary tables, but not much about the actual research methods involved, or how to turn the ISA model into something you can use to measure altitude in your own cool project. But hey, that's what this tutorial is for.

The ISA model assumes a whole host of things – that you're at the 45th parallel, that temperature at sea level is always 15° Celsius, etc… Some of these assumptions won't matter much, and others will. Just know that a good deal of all the tables and other info you might find on the net about altitude are also based on these assumptions. That's why I wasn't satisfied with reading the altitudes off some lookup-table on some web page – I wanted to know how to calculate this stuff myself, and that's what we're going to discuss now. Be prepared, because there's a bit o' math.

Atmospheric Temperature in Relation to Altitude

The best place to start is with the observation that as one travels upwards through the atmosphere, the temperature mostly seems to drop. We all know that if we go to the top of a mountain it's likely to be colder there than down in the valley, even on the same day.

Measurements have shown that the relationship between temperature and altitude are not consistent across the entire height of the atmosphere; however, they *are* consistent within atmospheric layers. Therefore, scientists split the atmosphere into various layers, and knowing how temperature behaves in each layer, arrived at an equation which relates temperature to altitude that will work for any layer- so long as you plug in the correct figures for the layer you're in. Let's look at this graphically first, though.

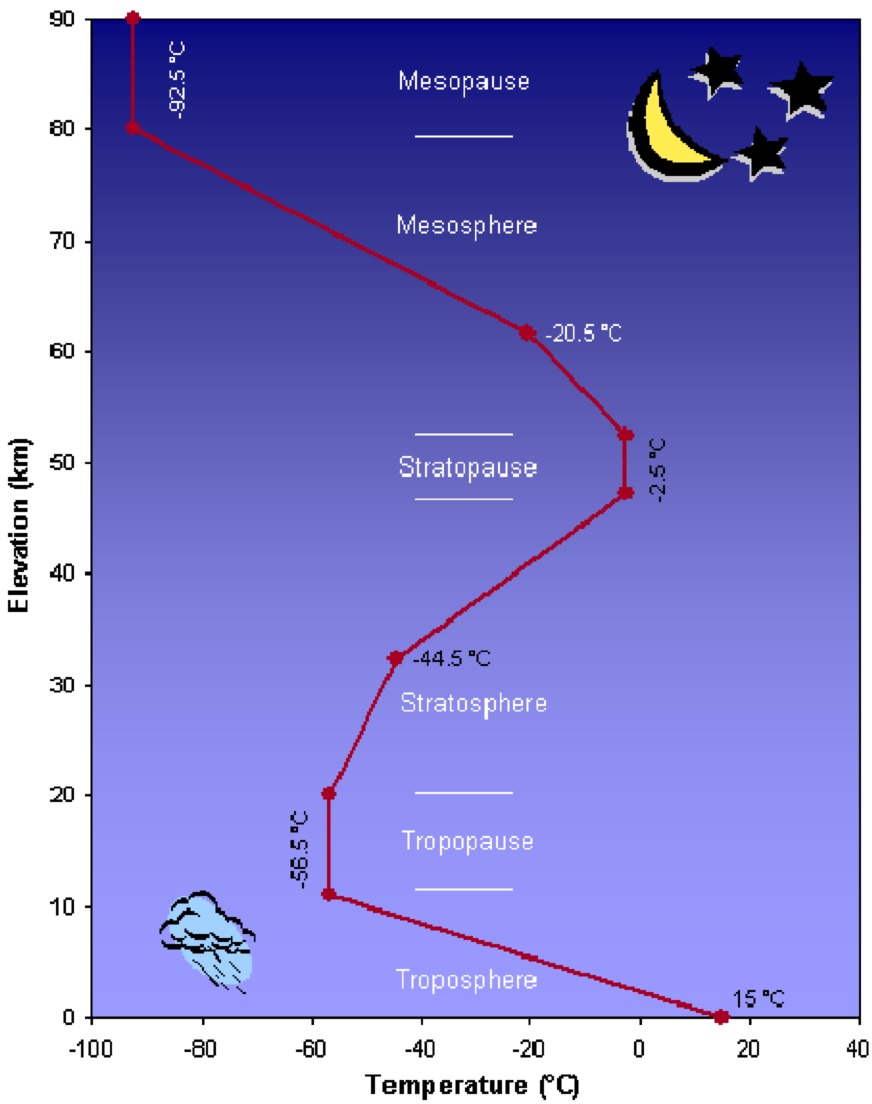

Figure 1 – Temperature change with respect to altitude

Figure 1 shows the change in measured temperature from sea level to approximately 90 km elevation. The layer adjacent to the surface of the earth is called the Troposphere and it extends to approximately 11 km. This is naturally the layer we live in, and also the layer in which the majority of our weather occurs.

Notice how in each layer the temperature behaves slightly differently: sometimes it decreases with altitude (as in the Troposphere), sometimes it remains steady (the "pause" layers), and in some layers it actually increases, such as in the layers of the Stratosphere – this is due to something with the ozone layer which I can't say I understand completely.

Of course, is the temperature within the Tropopause always -56.5 °C? Is it always 15 °C at sea level? No, we know these things fluctuate. The temperatures shown in Figure 1 are those assumed to be typical. They are what the ISA model calls the conditions on a "standard day". The standard day also includes assumptions about barometric pressure, air density and other things. Naturally we will want to adjust our measurements to reflect the conditions we experience on any given day that differ from the "standard day", and this is discussed more later. But for now let's assume that conditions in the atmosphere reflect those of the standard day.

Having seen the relationship between temperature and altitude graphically, let's now examine it mathematically. This is the ISA formula that relates temperature to altitude:

Eq. 1 $$T = T_1 + a(h - h_1)$$ - temperature equation

Where:

T	=	temperature we want to find
T_1	=	starting temperature of the atmospheric region or layer
a	=	the rate of temperature change in the given region – the slope of our line
h	=	altitude (height)
h_1	=	base height of the atmospheric region or layer

The equation could also be rearranged to solve for height, such that if we knew temperature we could estimate our altitude (though this is not the way altitude measurements are usually made).

The two variables T and h are self explanatory: one of them we must measure in order to solve for the other. The remaining three variables are given by the ISA model and they serve to describe the atmospheric layer of interest. T_1 is the temperature at the lower bound of the given layer, h_1 the starting altitude of the layer, and a the rate of change of temperature in the given layer. Table 1 gives the specific values of all three for each atmospheric layer up to the Mesopause (the atmosphere actually continues even higher, but we won't concern ourselves with it). The columns h_{upper} and T_{upper} are not used in the formula, but serve to illustrate the upper bound of the given layer.

Table 1
Defining Properties of the Standard Atmosphere

	Altitude (km)		Temperature (°C)		Slope (°C per km)
Region	h_1	h_{upper}	T_1	T_{upper}	a
Troposphere	0	11	15.0	-56.5	-6.5
Tropopause	11	20.1	-56.5	-56.5	0.0
Stratosphere	20.1	32.2	-56.5	-44.5	+1.0
Stratosphere	32.2	47.3	-44.5	-2.5	+2.8
Stratopause	47.3	52.4	-2.5	-2.5	0.0
Mesosphere	52.4	61.6	-2.5	-20.5	-2.0
Mesosphere	61.6	80	-20.5	-92.5	-4.0
Mesopause	80	~90	-92.5	-92.5	0.0

Let's work through a simple example so you can see how to plug the above numbers into the equation. Suppose we're flying along at an altitude of 40 km (roughly 130,000 feet- we're *high*). We want to know what the temperature is, and we don't want to stick our head outside to find out. Rather, we use Equation 1. We look at Table 1 to get T_1, h_1, and a for our specific layer, which at 40 km falls in the upper Stratosphere. Plugging this into the formula gives us:

Eq. 1 $$T = T_1 + a(h - h_1)$$

$$T = -44.5\,°C + 2.8\,\frac{°C}{km}\,(40\text{ km} - 32.2\text{ km})$$

$$T = -22.66\,°C$$

You can see that the kilometer symbols cancel out, leaving us with temperature. As a sanity check, we look at Table 1 and see if -22.66 °C is within the range of temperatures specified for the upper Stratosphere – and yes, it is.

Alright, so that's pretty basic stuff, but let's recap: the atmosphere can be subdivided into many layers like onion rings. In each of these layers temperature behaves in a consistent, predictable manner which can be described mathematically. The Standard Atmosphere model defines the several properties of each layer, such as the altitude at which layers begin and end, and how temperature varies within each. These property definitions are what researchers have found to be typical, average, or most-common conditions, and so make up what is known as the "standard day". Naturally real-life conditions will vary, but it helps to have a common framework to start from.

Knowing that, we can move on to another property of the atmosphere, pressure.

Atmospheric Pressure Explained

Although it appears that Equation 1 would allow us to determine altitude with a simple thermometer, in practice this is not nearly so accurate as measuring air pressure and it doesn't work at all in isothermal regions (atmospheric layers in which the temperature remains constant). Therefore, air pressure is the method used by almost every aircraft in the world to determine altitude - but don't worry, our temperature equation will still come in handy so you haven't wasted your time learning it.

It would help us first of all to have a basic understanding of what air pressure actually is. I'm not an atmospheric scientist or even a scientist of anything else really, but here's the best explanation I can come up with:

We know the earth has a gravitational force that pulls objects towards it. It pulls you and me towards it, which is why we can't leap over tall buildings. The mass of my body, being drawn to the surface of the earth by gravity, is going to result in an exertion of some amount of pressure on the ground that I could measure, say in pounds per square inch (psi). That pressure against the ground would be greater if I were a bigger person or maybe if I were a large building, whereas if I were a feather it would be less. The amount of pressure depends on my mass and the gravitational strength in the place where I am (we'll assume we remain on earth, so it should be constant).

The atmosphere behaves no differently from other objects like humans and bowling balls and feathers. Nitrogen, oxygen, helium and all the other sorts of elements that make up our atmosphere do have a mass (albeit small), and the earth's gravity pulls these things towards it, the result being that air exerts a pressure on the surface of the earth in the same way our feet do. But since the atmosphere is a gas it doesn't just exert pressure at the bottom of its "feet" the way a human does, it exerts an equal pressure in every direction. It exerts the same pressure on the surface of the earth as it does on your face and the walls of your building and everything else. This is air pressure.

Ok, so the logical next question is: how much pressure does the air exert? The answer is, it varies. But in the same way thermal properties of the atmosphere have been defined based on average conditions known as the "standard day", so too has atmospheric pressure. On a "standard day" the air pressure at *sea level* is equal to "one atmosphere" which has been precisely defined thusly, in a wide variety of pressure-measurement units:

1 Atmosphere (atm)	=	760 millimeters of mercury (mm. Hg)
	=	29.92 inches of mercury (in. Hg)
	=	101.325 kilopascals (kPa)
	=	101,325 pascals (Pa)
	=	1013.25 millibars (mb)
	=	14.696 pounds per square inch (psi)
	=	etc…

The final thing we need to know about air pressure is how it varies with altitude, because naturally it does. The higher you go up into the atmosphere, the less of the atmosphere there still is above you to weigh down on you, and therefore the air pressure is less. Another way to think of it is that the force of earth's gravity diminishes the farther away from the surface of it one goes. Since gravity is the prime source of air pressure (though not the only one), as gravity diminishes so does air pressure.

Unlike temperature, there are no layers of the atmosphere in which pressure remains constant. It varies at different *rates* in different regions, but it always varies - in fact it is such a fine measure that variations can be detected even in the space of a few centimeters or inches. This certainly can't be said for temperature, and it's another reason why air pressure is a more precise way to measure altitude.

If we had a formula that related air pressure to altitude, and we measured air pressure, we could then find our elevation. And of course this is what we'll now proceed to do, though we'll have to wade through some work to get there.

Atmospheric Pressure in Relation to Altitude

The ISA model doesn't actually give us a formula that relates pressure to altitude, at least one that's solved for us - instead it gives us one that relates pressure to atmospheric temperature. This is where our earlier temperature equation will come in handy, as we'll soon see. And secondly, it bears mentioning that the change in air pressure with respect to altitude is not linear but rather logarithmic, and what's more, the relationship differs depending on whether we're in an isothermal air layer or region (one where temperature remains constant), or in a non-isothermal region.

For these reasons, we're going to have to hold on to our neurons for the next page or so, as the formulas I'm going to unleash will consequently be a bit more involved than what we've seen heretofore.

This is the formula that relates air pressure to atmospheric *temperature* in the non-isothermal regions of the atmosphere (we'll deal with the isothermal formulas later):

Eq. 2 $$p = p_1 \left(\frac{T}{T_1} \right)^{-\frac{g}{aR}}$$ - pressure/temp. equation (non isothermal regions)

Where:

p	=	air pressure we want to find
p_1	=	starting pressure of the atmospheric region we're in
T	=	temperature we want to find
T_1	=	starting temperature of the atmospheric region or layer
g	=	acceleration due to gravity, assumed to be constant
a	=	the rate of temperature change in the given region – the slope of our line
R	=	ideal gas constant for air: also, naturally enough, assumed to be constant

Some of these variables should look familiar from our previous temperature formula, namely T, T_1, and a. The others should already start to make a little bit of sense from what we've discussed previously, but I'll touch on each of them here.

Note that we have a variable (actually a constant) for the force of gravity, *g*. This makes sense. The force of gravity is what draws us towards the surface of the earth, it's what gives us *weight*. Without gravity our atmosphere would quickly fly off into space. The fact that we do have gravity is not only what keeps our atmosphere hanging around, but also what causes it to exert a pressure commensurate with its mass, as discussed earlier. So it makes sense this formula should include the gravitational constant.

The "ideal gas constant" *R* isn't something you need to know much about, though if you want to do some reading about it there's plenty of interesting information you could learn. For now, it's enough to know that it deals in general with the mass of air, which we can also see would be important. To know how much pressure my feet are going to exert on the ground, we need to know how the force of gravity would act on my body's mass (as an aside, g*ravity* and *mass* are two concepts we often conveniently lump together with the term *weight*). To measure the pressure of the atmosphere we need to know the same sorts of things. Above with *g* we have information about gravity, and *R* is our information about the mass of the atmosphere (for you brainos out there, yes, I know I'm oversimplifying).

That leaves p and p_1 as the only other variables we've not yet discussed. The first is easy enough, p is the pressure of the air we are attempting to determine, or that we've measured. The second, p_1, is the pressure at the lower bound of the given atmospheric layer/region that we're in. You're probably sitting in the Troposphere right now. The pressure at the lower bound of that layer is therefore simply the pressure at sea level. Above the Troposphere is the Tropopause, and the pressure at the bottom of the Tropopause is the same as the pressure at the *top* of the layer below it (the Troposphere), and so on. It's the same concept as our T_1 discussed earlier in the temperature section.

Ok, so we have a dandy formula but it still doesn't give us altitude from air pressure. Also, it appears that it *tells* us what air pressure is, but we don't need to be told that because it's what we're going to be measuring with our cool pressure sensor. So let's rearrange Equation 2 like so:

Eq. 2-1 $$T = \left(\frac{p}{p1}\right)^{-\frac{aR}{g}} \cdot T_1$$ - solve Eq. 2 for T

Alright, that was quick and easy. Now, let's replace *T* with our earlier Equation 1 (I told you it would come in handy):

Eq. 1 $$T = T_1 + a(h - h_1)$$ - temperature equation

Eq. 2-2 $$T_1 + a(h - h_1) = \left(\frac{p}{p_1}\right)^{-\frac{aR}{g}} \cdot T_1$$ - substitute Eq. 1 into Eq. 2-1

Nice. Now we're getting somewhere. We have a formula that has both altitude (*h*) and pressure (*p*) in it. All we have to do is rearrange it so we solve for *h*. I'll save you a lot of weeping and gnashing of teeth and just show you the final result:

Eq. 3 $$h = \left(\frac{\left[\left(\frac{p}{p_1}\right)^{-\frac{aR}{g}} \cdot T_1\right] - T_1}{a}\right) + h_1$$ - Pressure altitude (non isothermal regions)

And voila! For a given air pressure measurement *p* we can now solve for our altitude *h*. All the other variables we know from Table 1, except for the two constants *R* and *g* (they're just constants so we'll worry about their actual values later), and p_1, which you can see in Table 2 below for each atmospheric layer.

Table 2
More Defining Properties of the Standard Atmosphere

	Altitude (km)		Pressure (Pa)		Density
Region	h_1	h_{upper}	p_1	p_{upper}	ρ_1
Troposphere	0	11	101,325	22,630.5	
Tropopause	11	20.1	22,630.5	5,474.27	
Stratosphere	20.1	32.2	5,474.27	867.849	
Stratosphere	32.2	47.3	867.849	110.874	
Stratopause	47.3	52.4	110.874	58.9822	
Mesosphere	52.4	61.6	58.9822	18.2033	
Mesosphere	61.6	80	18.2033	1.03719	
Mesopause	80	~90	1.03719	0.164295	

As before p_{upper} isn't used in any formula; I include it only to show you the range for the given region. Note also that although the pressures in Table 2 are listed in Pascals (Pa), the examples to follow will use kilopascals (kPa). To get kilopascals, simply divide Pascals by 1,000.

At this point we're pretty much through with our altitude formula. The Equation 3 we've just solved lets you plug in an air pressure reading and some constants, and out comes an altitude. We'll want to refine it a bit – such as an adjustment to compensate for conditions different than those on the so-called "standard day."

But before we go on to that, let's solve one more set of formulas. As mentioned earlier Equation 3 only applies in the layers of the atmosphere where temperature varies. This includes the Troposphere, Stratosphere, and Mesosphere. For the *isothermal* regions, where temperature remains constant, the formula is slightly different. Let's just run through it real quick-like.

Eq. 4 $$p = p_1 e^{-\left[\frac{g}{RT}\right](h - h_1)}$$ - pressure/temp. equation (isothermal regions)

Equation 4 is the equivalent of Equation 2 but for the isothermal regions of the atmosphere. Again we'll need to rearrange it, but one thing you'll notice that's different from Equation 2 is that height is already a part of the calculation, so we won't have to substitute in our Equation 1 (temperature/altitude equation). Another thing you'll want to notice is that although temperature

(*T*) is included, it will actually just be a constant – this is by definition what the temperature does in an isothermal region, it stays the same. To get this *T*, you'll just look up the value in Table 1.

So really all we need to do is rearrange Equation 4 so we have altitude (*h*) on the left side, and we're through. Again, I'll spare you all the bloody details and just show you the result (yes, the *ln* is a natural log):

Eq. 5

$$h = \left(\frac{ln\left[\frac{p}{p_1}\right]}{-\left[\frac{g}{RT}\right]} \right) + h_1$$

- Pressure altitude (isothermal regions)

And there you have it. With Equation 5 for isothermal regions and Equation 3 for the rest, and with the standard assumptions for some variables listed in Table 1 and 2, you have everything you need to determine altitude from air pressure. There are further refinements that we'll get to next, but these two formulas are the core of what you need to know - and if you don't plan on going higher than the Troposphere, which extends to 11 kilometers (36,000 feet) above sea level, then you really only need to know Equation 3.

Adjustments for Non-Standard Conditions

With Equation 3 or 5 and an air pressure measurement, we can now calculate our altitude at any location in the atmosphere. But what do we do when actual conditions aren't exactly the same as the standard day assumptions? Remember that the standard day defines the pressure at sea level, and this is used in all the calculations: for the Troposphere it is our sea level p_1, and the p_1 for all the other layers also depend on this initial assumption. If the sea level pressure today isn't quite the same as it is in our assumptions, our altitude measurements are going to be inaccurate. What we need is a way to compensate.

Fortunately this is not hard to do. We just need to know one thing, and that is our altitude at the place we want to calibrate our pressure sensor. You can get this by looking at a topo map or other source. TopoZone is a good place to go.

The process that has worked best for me is to find an alternative sea-level pressure that, if I used it in my formula, would give me the correct altitude for the air pressure I measure. The way I do this is re-solve my equations for sea level p_1, plugging in my known altitude and current air pressure reading, and using the resulting adjusted sea-level pressure instead of the standard day value.

This sounds a bit complicated, but it's not when you see it in action. Let's walk through it.

First, here's the original equation that relates pressure to temperature, just for review:

Eq. 2 $$p = p_1 \left(\frac{T}{T_1} \right)^{-\frac{g}{aR}}$$ - pressure/temp. equation (non isothermal regions)

Let's rearrange this so we have p_1 (pressure at sea-level) on the left side. Remember, p_1 isn't always the sea-level pressure, it's just the pressure at the lower bound of the atmospheric layer we happen to be dealing with. But since we're likely to be doing this calibration in the layer closest to earth (Troposphere), p_1 is equal to the pressure at sea-level.

Eq. 2-3 $$p_1 = p \left(\frac{T}{T_1} \right)^{\frac{g}{aR}}$$ - Eq. 2 solved for p_1

Now let's substitute our temperature equation, Equation 1, into Equation 2-3:

Eq. 1 $$T = T_1 + a(h - h_1)$$ - temperature equation

Eq. 6 $$p_1 = p \left(\frac{T_1 + ah}{T_1} \right)^{\frac{g}{aR}}$$ - substitute Eq. 1 into Eq. 2-3 (p_1 calibration for non-isothermal regions)

Now we can calculate an adjusted p_1 with Equation 6. For h we use our known altitude that we get from a topo map. For p we use the current air pressure as measured by our sensor. The rest of the items are constants. Once we have our new p_1, we don't need to calculate it again; we can just use it as a constant to replace the standard-day p_1 in our height formula (Equation 3). If we think we'll be traveling higher than the Troposphere, we'll also want to re-calculate the p_1's for the other regions as well.

Make sense? If not, don't worry. Next we'll do some examples with real live numbers, which should help.

Examples with Real Numbers

Let's review again quickly the variables and constants used in the following calculations. In all the examples that follow I'll be using metric units. Rather than get all confused with the standard measurement equivalents I'd suggest simply using metric throughout, and then at the final stage, if you'd rather have feet than meters, do the conversion there.

Variables	Description	Units
T	Current temperature	Kelvin
T1	Starting temperature of the atmospheric region or layer – see Table 1	Kelvin
h	Current altitude (technically known as pressure altitude)	meters
h1	Base height of the atmospheric region or layer – Table 1	meters
p	Current air pressure measurement	kPa
p1	Starting pressure of the atmospheric region – Table 2 *(note Table 2 units are Pa – so divide them by 1,000 to get kPa)*	kPa
a	The rate of temperature change in the given region – Table 1 (*no units – just use the number as given and remember the sign!)*	n/a

Constants	Description	Value
g	Acceleration due to gravity	*9.80665 m/sec*^{2}
R	Ideal gas constant	*287.05 Joule/kg/K°*

Example 1
Assume standard-day conditions. Current air pressure is 90 kPa. What is the altitude?

Solution
The first thing to determine is which altitude formula to use – remember there are two formulas, one that applies to isothermal regions of the atmosphere and the other for non-isothermal (Equations 5 and 3, respectively). An easy way for a human to figure this out is to look at Table 1, from which we can tell that 90 kPa falls in the Troposphere. Therefore we use Equation 3 to solve this problem:

Eq. 3
$$h = \left(\frac{\left[\left(\frac{p}{p_1} \right)^{-\frac{aR}{g}} \cdot T_1 \right] - T_1}{a} \right) + h_1$$
- Pressure altitude (non-isothermal)

From Table 1 we will also want to get the values of T_1, p_1 and h_1. Some notes:

- Since we're in the Troposphere, which is the lowest atmospheric layer, the bottom height h_1 is just going to be zero.
- Remember that the formula requires temperatures to be in Kelvin. To convert Celsius to Kelvin simply add 273.15. We need to do this to the value of T_1 that we use from Table 1.
- Table 1 also lists *a* as the change in degrees Celsius per kilometer. However, this formula is dealing with meters – therefore, we must first divide the *a* in Table 1 by 1,000 before using it – this converts it to the "change in degrees Celsius per meter."

Plugging in the numbers gives us:

$$h = \frac{\left[\left(\frac{90}{101.325} \right)^{-\frac{\left(\left[\frac{-6.5}{1000} \right] * 287.05 \right)}{9.80665}} * (15 + 273.15) \right] - (15 + 273.15)}{\left(\frac{-6.5}{1000} \right)} + 0$$

h = 988.5 meters (3,242 feet)

This is the altitude corresponding to a pressure reading of 90 kPa, assuming standard-day conditions prevail.

Example 2
You happen to know that the altitude of the location you are standing at is 988.5 meters above sea level (the altitude we just computed – obviously you live on a mountain). However, your sensor tells you that current air pressure is 91.035 kPa. From Example 1 above we know that on a "standard day" your pressure sensor should actually give you a reading of exactly 90 kPa.

Part 1: If you don't adjust, what altitude will the standard assumptions tell you you're at?

Part 2: What should p_1 be adjusted to so that the altitude calculation correspondents with reality?

Solution to Part 1
The process here is the same as in our first example. Let's just plug in the air pressure reading and solve for altitude.

$$h = \frac{\left[\left(\frac{91.035}{101.325}\right)^{-\frac{\left(\left[\frac{-6.5}{1000}\right] * 287.05\right)}{9.80665}} * (15 + 273.15)\right] - (15 + 273.15)}{\left(\frac{-6.5}{1000}\right)} + 0$$

Solving the equation tells us that our present altitude is 894 meters. If we really are sure that our actual altitude is 988.5 meters, then you can see this calculation is rather off – by almost 100 meters or about 300 feet.

Solution to Part 2
What we need to do to get an accurate altitude reading is adjust our *p1* (or, since we're dealing with the Troposphere, our sea level pressure), so that it reflects the conditions of our present day, rather than the "standard day" assumptions. To do this we recall Equation 6.

Eq. 6 $$p_1 = p\left(\frac{T_1 + ah}{T_1}\right)^{\frac{g}{aR}}$$ - p_1 calibration (non-isothermal regions)

Let's plug in what we know – namely our pressure reading of 91.035 kPa and the thing that makes this formula different: our known altitude *h*. But again, some notes:

- First of all, I've dispensed with adding 273.15 to my Celsius temperatures to arrive at Kelvin. Instead, I wrote the Kelvin value directly (15° Celsius + 273.15 = 288.15)

- I've also divided *a* by 1,000 beforehand rather than writing it out (remember we need to convert the rate of change from degrees Celsius per kilometer in Table 1 to degrees Celsius per *meter* – hence an *a* in the Troposphere of -6.5 becomes -0.0065)

$$p_1 = 91.035 * \left(\frac{288.15 + [-0.0065 * 988.5]}{288.15} \right)^{\frac{9.80665}{(-0.0065 * 287.05)}}$$

$$p_1 = 102.49 \text{ kPa}$$

And there we have it. Now, instead of using 101.325 as our p_1 in the Troposphere, which is the standard day assumption (from Table 2), we should use the value we just solved for. Using it will give us our actual altitude.

Want to make sure it really works? Let's redo the Solution to Part 1, only using this new p_1. Remember that our sensor is giving us an air pressure reading of 91.035 kPa, and we know from a topographical map or some other source that our actual altitude is 988.5 meters. In Example 1 we determined that in fact at an altitude of 988.5 meters the air pressure should instead be an even 90 kPa. If we use the unadjusted altitude formula (Equation 3), as we did in Part 1 of this problem, the resulting altitude calculation is 894 meters, which is less than what we know our true altitude to be. So here we re-solve Part 1 only we use our adjusted p_1 instead of the standard day value (it's in bold).

$$h = \frac{\left\{ \left(\left[\frac{91.035}{\mathbf{102.49}} \right]^{-\frac{\left[-\left(\frac{6.5}{1000} \right) * 287.05 \right]}{9.80665}} \right) * (15 + 273.15) \right\} - (15 + 273.15)}{-0.0065} + 0$$

h = 988.5 meters (3,242 feet) *It works!*

This is the same altitude that we solved for in Example 1, even though the pressure reading is different. We get the same altitude this time because we used a p_1 adjusted for current conditions.

Example 3
Calculate your altitude at an air pressure reading of 15 kPa. However, don't assume standard day conditions – instead, use the adjustment for p_1 that you calculated in Example 2.

Solution
Note first of all (from Table 2) that 15 kPa puts us well into the Tropopause. Still, the altitude calculation in this problem is not difficult – we simply need to plug what we know into Equation 5, which is our altitude formula for isothermal regions (by definition, the Tropo*pause* is an isothermal region: the temperature remains constant within it).

What's tricky about this problem is that it requires us to solve for something not yet explicitly covered in this tutorial. Recall that p_1 is the air pressure at the bottom of a given air layer or atmospheric region. For the Troposphere, which is what we've been dealing with thus far, p_1 is equal to the pressure at sea level. But this is not the case for the Tropopause, where p_1 is essentially equal to the air pressure at the very top of the layer immediately beneath it (the Troposphere). Table 2 gives us values for p_1 in every air layer, but those are standard day assumptions. We need to calculate a new $p_{1\_tropopause}$. The question says "use the adjustment for p_1 that you calculated in Example 2," yet in Example 2 we didn't calculate an adjusted $p_{1\_tropopause}$, we calculated an adjusted $p_{1\_troposphere}$. Nevertheless, the question isn't written incorrectly: to calculate an adjusted $p_{1\_tropopause}$, we'll still need to use the adjusted $p_{1\_troposphere}$ that we found in Example 2.

There is more than one way to do this, but the way I would go about it is to return all the way back to our Equation 2, which relates pressure to temperature in non-isothermal regions, such as the Troposphere:

Eq. 2 $$p = p_1 \left(\frac{T}{T_1} \right)^{-\frac{g}{aR}}$$ - pressure/temp. equation (non isothermal)

We want to find the pressure at the base of the Tropopause, what I'm calling $p_{1\_tropopause}$, and to do so I'll use the trick that this value is going to be equal to the pressure at the very top of the layer beneath it - the Troposphere. So in fact this will not be that hard. For T and T_1 we use the temperature values for the top and bottom of the Troposphere, respectively, as shown in Table 1. For p_1 we use our adjusted p_1 that we calculated in Example 2. The rest are constants, except for a, which we get from Table 1

$$p = 102.49 * \left(\frac{[-56.5 + 273.15]}{[15 + 273.15]} \right)^{-\frac{9.80665}{(-0.0065 * 287.05)}}$$

$$p = 22.8919 = p_{1\_tropopause}$$

Now that we have our adjusted p_1 for the Tropopause to replace the standard-day value, we're all set to calculate altitude. For this we use Equation 5, which is the altitude formula for isothermal regions. For p_1 we use our $p_{1\_tropopause}$ and for p we use what was given to us in the problem, 15 kPa. For h_1 we'll use 11,000 meters – this is the altitude at the base of the Tropopause. And finally for T we use the temperature listed in Table 1 for the Tropopause, -56.5° Celsius, but of course we convert it to Kelvin first. The rest is just the standard stuff.

Eq. 5

$$h = \left(\frac{ln\left[\frac{p}{p_1}\right]}{-\left[\frac{g}{RT}\right]} \right) + h_1$$

- Pressure altitude (isothermal regions)

$$h = \left(\frac{ln\left[\frac{15}{22.8919}\right]}{-\left[\frac{9.80665}{287.05 * 216.65}\right]} \right) + 11{,}000$$

h = 13,681 meters (44,885 feet)

We've now measured the altitude corresponding to a pressure reading of 15 kPa, which happens to be in the Tropopause, and we even calculated and used an adjusted $p_{1\_tropopause}$. Good job.

For what it's worth, if we'd used the standard-day assumption for $p_{1\_tropopause}$ rather than our adjusted value, the altitude would have ended up being around 13,610 meters. Not a whole lot different, but if you're going to do something, you might as well do it right.

Conclusion

And with those examples I hope you now have a fairly thorough understanding of the math behind altitude measurements based on atmospheric pressure. I've made up some functions to do a lot of this stuff in the *C* programming language, which you can feel free to use in your own projects.